

ON A MULTISCALE ANALYSIS OF A MICRO-MODEL OF HEAT TRANSFER IN BIOLOGICAL TISSUES

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ABSTRACT. A bio-heat transfer model for biological tissues in a micro-scale and periodical settings is investigated . It is assumed that the model is a two-component system consisting of solid particles representing tissue cells and interconnected pores containing either arterial or venous blood. This tissue-blood system is described by two energy equations, one equation for the solid tissues and the other for the surrounding blood. On the interface between them, it is assumed that the heat transfer is governed by Newton's cooling law. Using homogenization techniques, it is shown that the obtained macro-model presents some extra-terms and it can be seen as a new mathematical model for human thermotherapy and human thermoregulation system.

1. INTRODUCTION

Studying heat transfer in biological tissues is important in many biomedical engineering such as thermoregulation system, thermotherapy and radiotherapy, skin surgery etc... See for instance S.A. Berger & *al.* [11], J.C. Chato [13], M. Gautherie [17], K. Khanafer & *al.* [21], M. Miyikawa and J.C. Bolomey (eds) [22] and the references therein. Many mathematical models were proposed to predict the distribution of the temperature in biological tissues. One of the most widely used model is the bioheat equation after the pioneering work of H.H. Pennes [23]. It is based on the well-known Fourier law with the concept of blood perfusion. It reads as follows

$$\rho c \partial_t T - \operatorname{div}(\kappa \nabla T) + \omega_b \rho_b c_b \rho (T - T_a) = f \quad (1.1)$$

where T, ρ, c and κ are respectively the temperature, the density, the specific heat and the heat conductivity coefficient of the tissue, ω_b is the blood perfusion, ρ_b and c_b are the density and the specific heat of

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the blood and T_a is the temperature of the arterial flow. Finally f is some external source of heating and it is generally written as the sum of sources due to absorbed laser light and metabolic activity. Many scientists have attempted to justify the Helmholtz term of (1.1). We mention for example M.M. Chen, K.R. Holmes[14], P. Wust & *al.* [27], R. Hochmuth and P. Deuffhard[18]. In the latter, a homogenization technique was developed on a microvascular model consisting of tissues (solid) surrounded by blood (fluid). More precisely they study the following system

$$-\Delta T^\varepsilon = S^\varepsilon \text{ in } \Omega^\varepsilon, \quad (1.2)$$

$$T^\varepsilon = 0 \text{ on } \partial\Omega, \quad (1.3)$$

$$\frac{\partial T^\varepsilon}{\partial n^\varepsilon} = \varepsilon \alpha (T_b^\varepsilon - T^\varepsilon) \text{ on } \partial Q^\varepsilon \quad (1.4)$$

where T^ε is the temperature and S^ε the source term in the tissues represented by Ω^ε whereas T_b^ε is the temperature of the blood. The domain Ω^ε is obtained by removing from Ω , a bounded and regular domain, a set of holes Q^ε where blood flows. In (1.4), n^ε is the unit normal of ∂Q^ε outward to Ω^ε and $\alpha > 0$ a physiological parameter. They obtained the following homogenized model:

$$-\operatorname{div}(\mathcal{A}\nabla T^*) + \alpha^*(T^* - T_b) = S$$

where \mathcal{A} is the homogenized tensor, T^* is the weak limit in $H^1(\Omega)$ of some extended temperature $P^\varepsilon(T^\varepsilon)$, S^ε is the weak limit in $L^2(\Omega)$ of the source term S^ε and finally α^* is the effective Helmholtz term. In fact, biological tissues can be seen as porous media where cells (matrix) are separated by voids or pores which are filled with blood. This system of cells-blood can be interpreted as a two-constituent medium. In connection with binary composites presenting thermal barriers at the interfacial contact, we mention especially the work by J.L. Auriault and H. Ene[9] where they study heat transfer in a two-component composite with conductivities of the same order of magnitude. The macroscopic model is shown to belong to two main types of field models: one-temperature and two-temperature, depending on the order of magnitude of the interfacial thermal conductance. In the present paper, we shall be concerned with a micro-model for the heat transfer in a biological tissue made of two interacting systems (cells tissues and blood regions) where the conductivities are assumed to be of different order of magnitude. We also assume that the transition between these two regions on the interface is governed by Newton's cooling law. That is the heat flow through the interface is proportional, by the thermal conductance of the layer, to the jump of the temperature field, see

R. Hochmuth and P. Deuffhard[18] (see also H.S. Carslaw and J.C. Jaeger[12]). We mention that this kind of boundary transmission condition was used for the homogenization in porous media, see for e.g. A. Ainouz[2],[3],[4]. In fact, there are many works showing how transport theories in porous media enhance the understanding of flow and heat transfer in biological tissues. For more details, we refer the reader to the survey paper by A.-R.A. Khaled and K. Vafai[20].

The paper is organized as follows: in Section 2, the geometry of the domain and the micro-model are set. In Section 3 a formal expansion technique is used to derive the homogenized model. Finally in Section 4, the two-scale convergence technique is applied to justify the formal procedure of Section 3.

2. SETTING OF THE PROBLEM

We start by introducing the notation used throughout this paper. We consider Ω a bounded domain in \mathbb{R}^d ($d \geq 2$) with smooth boundary $\partial\Omega$. Let $Y \stackrel{\text{def}}{=}]0, 1[^d$ be the generic cell of periodicity divided as $Y = Y_1 \cup Y_2 \cup \Sigma$ where Y_1, Y_2 are two connected, open disjoint subsets of Y and $\Sigma \stackrel{\text{def}}{=} \partial Y_1 \cap \partial Y_2$ is a smooth $(d-1)$ -dimensional manifold. As in G. Allaire and F. Murat[7], we assume that the Y -periodic continuation of Y_1 to the whole space \mathbb{R}^d , namely $\tilde{Y}_1 = \cup_{k \in \mathbb{Z}^d} (k + Y_1)$ is smooth and connected. Note that no connectedness assumption is made on the part $\cup_{k \in \mathbb{Z}^d} (k + Y_2)$.

Let χ_1 (resp. χ_2) denote the Y -periodic characteristic function of Y_1 (resp. Y_2). Denoting $\varepsilon > 0$ a sufficiently small parameter, we set

$$\Omega_1^\varepsilon \stackrel{\text{def}}{=} \{x \in \Omega : \chi_1(\frac{x}{\varepsilon}) = 1\}, \quad \Omega_2^\varepsilon \stackrel{\text{def}}{=} \{x \in \Omega : \chi_2(\frac{x}{\varepsilon}) = 1\},$$

and let $\Sigma^\varepsilon \stackrel{\text{def}}{=} \overline{\Omega_1^\varepsilon} \cap \overline{\Omega_2^\varepsilon}$. Without loss of generality, we assume that the region Ω_2^ε is strictly embedded in the region Ω_1^ε , in the sense that $\overline{\Omega_2^\varepsilon} \subset \Omega$. In this connection, Ω_1^ε is referred as the cellular domain and Ω_2^ε as the voids filled with blood. We see that the boundary of Ω_2^ε is the interface Σ^ε and the boundary of Ω_1^ε consists then of two parts: Σ^ε and the exterior boundary Γ . We can write that

$$\partial\Omega_2^\varepsilon = \Sigma^\varepsilon \text{ and } \partial\Omega_1^\varepsilon = \partial\Omega \cup \Sigma^\varepsilon.$$

Thanks to the connectedness of \tilde{Y}_1 , we see that Ω_1^ε is connected while Ω_2^ε may or may not be connected.

Let us denote ρ, c and κ the density, the specific heat and the heat conductivity coefficient of the tissue, respectively. Let ω_b denote the blood perfusion, ρ_b and c_b the density and the specific heat of the

blood. Let κ_b the heat conductivity coefficient of the blood. We shall assume that the phenomenological parameters: $\rho, c, \kappa, \omega_b, \rho_b, c_b$ and κ_b are positive constant and independent of ε .

Let $(0, T)$ be the time interval. Put

$$\begin{aligned} Q &\stackrel{\text{def}}{=} (0, T) \times \Omega, \quad \Gamma \stackrel{\text{def}}{=} (0, T) \times \partial\Omega, \quad S \stackrel{\text{def}}{=} (0, T) \times \Sigma, \\ Q_1^\varepsilon &\stackrel{\text{def}}{=} (0, T) \times \Omega_1^\varepsilon, \quad Q_2^\varepsilon \stackrel{\text{def}}{=} (0, T) \times \Omega_2^\varepsilon, \quad S^\varepsilon \stackrel{\text{def}}{=} (0, T) \times \Sigma^\varepsilon \end{aligned}$$

Let also

$$\alpha = \frac{\kappa}{\rho c}, \quad \alpha_b = \frac{\kappa}{\rho_b c_b}, \quad \alpha_b^\varepsilon = \varepsilon^2 \alpha_b, \quad \gamma = \omega_b \rho_b \frac{c_b}{c}. \quad (2.1)$$

The micro-model that we shall study in this paper is as follows:

$$\partial_t T^\varepsilon - \alpha \Delta T^\varepsilon = F \quad \text{in } Q_1^\varepsilon, \quad (2.2a)$$

$$\partial_t T_b^\varepsilon - \alpha_b^\varepsilon \Delta T_b^\varepsilon = F_b \quad \text{in } Q_2^\varepsilon, \quad (2.2b)$$

$$\alpha \nabla T^\varepsilon \cdot \nu^\varepsilon = \alpha_b^\varepsilon \nabla T_b^\varepsilon \cdot \nu^\varepsilon \quad \text{on } S^\varepsilon, \quad (2.2c)$$

$$\alpha \nabla T^\varepsilon \cdot \nu^\varepsilon = -\varepsilon \gamma (T^\varepsilon - T_b^\varepsilon) \quad \text{on } S^\varepsilon, \quad (2.2d)$$

$$T^\varepsilon = 0 \quad \text{on } \Gamma, \quad (2.2e)$$

$$T^\varepsilon(0, \cdot) = h(\cdot) \quad \text{in } \Omega_1^\varepsilon, \quad (2.2f)$$

$$T_b^\varepsilon(0, \cdot) = h_b(\cdot) \quad \text{in } \Omega_2^\varepsilon \quad (2.2g)$$

where f (resp. f_b) be some external source of heating in cells (resp. blood), ν^ε stands for the unit normal of Σ^ε outward to Ω_1^ε and h, h_b are the initial temperature field in $\Omega_1^\varepsilon, \Omega_2^\varepsilon$ respectively. Without no loss of generality, we shall assume

$$f, f_b, h, h_b \in L^2(\Omega). \quad (2.3)$$

Our system actually models heat flow in a porous medium (biological tissue). In this connection, Ω_1^ε represents the matrix-cells space region and Ω_2^ε the pores which are filled with blood. The thin layer Σ^ε is an interfacial flow barrier with heat conductance given by $\gamma^\varepsilon = \varepsilon \omega_b \rho_b c_b \rho$. The unknowns T^ε and T_b^ε are the temperatures in Q_1^ε and Q_2^ε respectively. The first equation describes the heat flow in the cells with large conductivity and the second describes the heat flow in the blood region with low conductivity. Condition (2.2c) expresses flux continuity across the interface. However, the temperature may present in general jumps across Σ^ε . Here, we have employed Newton's cooling law described by (2.2d), see for instance R. Hochmuth and P. Deuffhard[18] and H.S. Carslaw and J.C. Jaeger[12]. We can say that Ω_1^ε can be considered as a good conductor, while Ω_2^ε a poor one. The interface Σ^ε can be seen as a heat exchanger. The condition (2.2e) is the standard homogeneous Dirichlet condition on the exterior boundary. Finally equations

(2.2f)-(2.2g) are the initial conditions, which close the system under consideration. Note that in (2.1), the matrix A_b^ε is scaled by ε^2 to provide the correct scaling for the heat flow in the block regions. Indeed, this scaling is the unique choice that makes every term of the porous medium equation in the block cell reappears in the leading order asymptotic expansion, so that the form of the equation is preserved on the small scale independently of ε .

Let us now establish a variational framework of our problem. To this end, we first introduce some notations. Let

$$H^\varepsilon = L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon) \text{ and } V^\varepsilon = (H^1(\Omega_1^\varepsilon) \cap H_0^1(\Omega)) \times H^1(\Omega_2^\varepsilon).$$

We shall consider on H^ε and V^ε the following inner products:

$$\begin{aligned} (\varphi, \psi)_{H^\varepsilon} &= \int_{\Omega_1^\varepsilon} \varphi_1 \psi_1 dx + \int_{\Omega_2^\varepsilon} \varphi_2 \psi_2 dx, \quad \varphi = (\varphi_1, \varphi_2), \quad \psi = (\psi_1, \psi_2), \\ (\varphi, \psi)_{V^\varepsilon} &= \int_{\Omega_1^\varepsilon} \nabla \varphi_1 \nabla \psi_1 dx + \varepsilon^2 \int_{\Omega_2^\varepsilon} \nabla \varphi_2 \nabla \psi_2 dx + \\ &\quad \varepsilon \int_{\Sigma^\varepsilon} (\varphi_1 - \varphi_2) (\psi_1 - \psi_2) d\sigma^\varepsilon \end{aligned}$$

where dx denotes the Lebesgue measure on \mathbb{R}^d and $d\sigma^\varepsilon$ the surfacic measure on Σ^ε . The norms induced in H^ε and V^ε are denoted by $\|\cdot\|_{H^\varepsilon}$ and $\|\cdot\|_{V^\varepsilon}$, respectively. Clearly, H^ε and V^ε are Hilbert spaces when equipped with their respective norms. Moreover, it can easily be shown that V^ε is separable, dense and continuously embedded in H^ε .

Let us introduce the bilinear form $a^\varepsilon(\cdot, \cdot) : V^\varepsilon \times V^\varepsilon \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} a^\varepsilon(\varphi, \psi) &= \int_{\Omega_1^\varepsilon} \alpha \nabla \varphi_1 \nabla \psi_1 dx + \int_{\Omega_2^\varepsilon} \alpha_b \nabla \varphi_2 \nabla \psi_2 dx + \\ &\quad \int_{\Sigma^\varepsilon} \gamma^\varepsilon (\varphi_1 - \varphi_2) (\psi_1 - \psi_2) d\sigma^\varepsilon \end{aligned}$$

where $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2) \in V^\varepsilon$. We see that $a^\varepsilon(\cdot, \cdot)$ is continuous and uniformly coercive.

Let $(V^\varepsilon)'$ denote the dual space of V^ε . Let $\mathcal{A}^\varepsilon \in \mathcal{L}(V^\varepsilon, (V^\varepsilon)')$ be given by

$$\mathcal{A}^\varepsilon(\varphi)\psi = a^\varepsilon(\varphi, \psi), \quad \varphi, \psi \in V^\varepsilon.$$

For convenience we shall denote $w^\varepsilon = (T^\varepsilon, T_b^\varepsilon)$, $g = (h, h_b)$ and

$$f^\varepsilon = f\chi_1^\varepsilon + f_b\chi_2^\varepsilon, \quad \chi_m^\varepsilon(x) = \chi_m\left(\frac{x}{\varepsilon}\right), m = 1, 2.$$

Let

$$W^{1,2}(0, T; H^\varepsilon) = \left\{ w \in L^2(0, T; H^\varepsilon) : w' = \frac{dw}{dt} \in L^2(0, T; H^\varepsilon) \right\}.$$

The variational formulation for (2.2a)-(2.2g) reads as follows: find $w^\varepsilon \in L^2(0, T; V^\varepsilon)$ such that for every $\varphi \in W^{1,2}(0, T; H^\varepsilon) \cap L^2(0, T; V^\varepsilon)$ with $\varphi(T) = 0$, we have

$$\begin{aligned} & - \int_0^T (w^\varepsilon(t), \partial_t \varphi(t))_{H^\varepsilon} dt + \int_0^T \mathcal{A}^\varepsilon(w^\varepsilon(t)) \varphi(t) dt \\ & = \int_0^T (f^\varepsilon, \varphi(t))_{H^\varepsilon} dt + (g, \varphi(0))_{H^\varepsilon}. \end{aligned} \quad (2.4)$$

where dt denotes the Lebesgue measure on $(0, T)$.

Next, we state the existence and uniqueness result for (2.4) the proof of which is given in the next section.

Theorem 1. *Let $\varepsilon > 0$ be a sufficiently small parameter. Then, there exists a unique weak solution $w^\varepsilon \in L^2(0, T; V^\varepsilon)$ of Problem (2.4) and the following energy estimate holds:*

$$\|w^\varepsilon\|_{L^\infty(0, T; H^\varepsilon)} + \|w^\varepsilon\|_{L^2(0, T; V^\varepsilon)} \leq C. \quad (2.5)$$

Now, we are ready to give the main result of this paper whose proof will be given in the last section.

We define the overall temperature in the biological tissue region $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$ by

$$u^\varepsilon(t, x) = \chi_1\left(\frac{x}{\varepsilon}\right) T^\varepsilon(t, x) + \chi_2\left(\frac{x}{\varepsilon}\right) T_b^\varepsilon(t, x), \quad \text{a.e. } (t, x) \in Q.$$

Theorem 2. *There exists a subsequence of (u^ε) , still denoted (u^ε) such that there exist $T \in L^2(0, T; H_0^1(\Omega))$ and $T_b \in L^2(Q; H_\#^1(Y))$ with*

- (1) $\chi_1\left(\frac{x}{\varepsilon}\right) u^\varepsilon$ weakly converges to $|Y_1| T$ in $L^2(Q)$;
- (2) $\chi_2\left(\frac{x}{\varepsilon}\right) u^\varepsilon$ weakly converges to $\int_{Y_2} T_b dy$ in $L^2(Q)$;
- (3) T is a solution to the homogenized problem:

$$\partial_t T - \int_0^t \mathcal{H}(t - \tau) T(\tau) d\tau - \operatorname{div}(\tilde{A} \nabla T) + \tilde{\gamma} T = \mathcal{F} \text{ in } Q, \quad (2.6)$$

$$T = 0 \text{ on } S, \quad (2.7)$$

$$T(0, x) = |Y_1| h(x), \quad x \in \Omega \quad (2.8)$$

where \mathcal{H} , \tilde{A} , $\tilde{\gamma}$ and \mathcal{F} are respectively given by (4.24), (4.17), (4.23) and (4.25);

- (4) The temperature T_b is related to T by:

Finally, we end this section by noticing that (2.6) is an integro-differential equation of Barbashin type.

3. SOME AUXILIARY LEMMAS AND PROOF OF THEOREM 1

We begin this section with some standard lemmas needed for proving the existence and uniqueness result and also for establishing uniform a priori estimates that are specifically important when using compactness techniques .

Lemma 1. *There exists a constant $C > 0$, independent of ε such that for all $\varphi_1 \in H_0^1(\Omega) \cap H^1(\Omega_1^\varepsilon)$ we have*

$$\|\varphi_1\|_{0,\Omega_1^\varepsilon} \leq C \|\nabla \varphi_1\|_{0,\Omega_1^\varepsilon}. \quad (3.1)$$

Proof. See for instance G. Allaire and F. Murat[7, Lemma A.4]. \square

Lemma 2. *There exists a constant $C > 0$, independent of ε such that for all $\varphi_2 \in H^1(\Omega_2^\varepsilon)$ we have*

$$\|\varphi_2\|_{0,\Omega_2^\varepsilon}^2 \leq C \left(\varepsilon^2 \|\nabla \varphi_2\|_{0,\Omega_2^\varepsilon}^2 + \varepsilon \|\varphi_2\|_{0,\Sigma^\varepsilon}^2 \right). \quad (3.2)$$

Proof. See C. Conca[16, Lemma 6.1]. \square

Lemma 3. *There exists a constant $C > 0$, independent of ε such that for all $\varphi \in H^1(\Omega_1^\varepsilon)$ we have*

$$\varepsilon \|\varphi\|_{0,\Sigma^\varepsilon}^2 \leq C \left(\varepsilon^2 \|\nabla \varphi\|_{0,\Omega_1^\varepsilon}^2 + \|\varphi\|_{0,\Omega_1^\varepsilon}^2 \right), \quad (3.3)$$

and

$$\varepsilon \|\varphi\|_{0,\Sigma^\varepsilon}^2 \leq C \left(\|\nabla \varphi\|_{0,\Omega_1^\varepsilon}^2 \right). \quad (3.4)$$

Proof. Using the trace theorem on Y_1 (see for e.g. R. A. Adams and J. F. Fournier[1]), we know that there exists a constant $C(Y_1) > 0$ such that for every $\psi \in H^1(Y_1)$

$$\int_{\Sigma} |\psi|^2 d\sigma \leq C \left(\int_{Y_1} |\nabla \psi|^2 dy + \int_{Y_1} |\psi|^2 dy \right).$$

Then, we change y by x/ε and we get

$$\varepsilon \int_{\Sigma^{\varepsilon k}} |\varphi|^2 d\sigma^\varepsilon \leq C \left(\varepsilon^2 \int_{Y_1^{\varepsilon k}} |\nabla \varphi|^2 dx + \int_{Y_1^{\varepsilon k}} |\varphi|^2 dx \right), \quad (3.5)$$

for every $\varphi \in H^1(Y_1^{\varepsilon k})$, where

$$Y_1^{\varepsilon k} = \varepsilon(k + \Omega_1^\varepsilon), \quad \Gamma^{\varepsilon k} = \varepsilon(k + \Gamma) \quad k \in \mathbb{Z}^d.$$

Note that the constant C appearing in (3.5) is the same for all $\varepsilon > 0$ and for all $k \in \mathbb{Z}^d$. Now, by taking the sum of the inequalities (3.5) over all the cells $Y_1^{\varepsilon k}$ contained in Ω , we obtain (3.3). In fact, on the part of the cells which contain a portion of the exterior boundary, the estimate (3.3) still holds true, since those cells lie at a distance $O(\varepsilon)$. As ε is sufficiently small, say $\varepsilon < 1$, we have from (3.3) that for all $\varphi \in H^1(\Omega_1^\varepsilon)$

$$\varepsilon \int_{\Gamma^\varepsilon} |\varphi|^2 d\sigma^\varepsilon \leq C \left(\int_{\Omega_1^\varepsilon} |\nabla \varphi|^2 dx + \int_{\Omega_1^\varepsilon} |\varphi|^2 dx \right) \quad (3.6)$$

and using the Friedrich inequality (3.1) in (3.6), we get (3.4). \square

Proof of Theorem 1. We shall use the Lions Lemma (see for instance R. Showalter[25, Prop. 2.3., Chap. III]). Since $a^\varepsilon(\cdot, \cdot)$ is coercive and continuous, it only remains to prove the continuity of the form:

$$\varphi = (\varphi_1, \varphi_2) \mapsto L^\varepsilon((\varphi_1, \varphi_2)) = \int_0^T (h^\varepsilon, \varphi(t))_{H^\varepsilon} dt$$

on $L^2(0, T; (V^\varepsilon)')$. First, using Cauchy-Schwarz inequality and (2.3), we see that for all $\varphi = (\varphi_1, \varphi_2) \in V^\varepsilon$,

$$\begin{aligned} |L^\varepsilon((\varphi_1, \varphi_2))| &= \left| \int_0^T \left(\int_{\Omega_1^\varepsilon} f_1 \varphi_1 dx + \int_{\Omega_2^\varepsilon} f_2 \varphi_2 dx \right) \right| \\ &\leq M(f_1, f_2) (\|\varphi_1\|_{0, \Omega_T^\varepsilon} + \|\varphi_2\|_{0, \Omega_T^\varepsilon}) \end{aligned} \quad (3.7)$$

where

$$M(f_1, f_2) = \max(\|f_1\|_{0, \Omega_T}, \|f_2\|_{0, \Omega_T}) < +\infty$$

is a constant independent of ε . Observe that $f \in L^2(0, T; H^\varepsilon)$. Next, from (3.2), we get

$$\begin{aligned} \int_{\Omega_2^\varepsilon} |\varphi_2|^2 dx &\leq C \left(\varepsilon^2 \int_{\Omega_2^\varepsilon} |\nabla \varphi_2|^2 dx + \varepsilon \int_{\Gamma^\varepsilon} |\varphi_1 - \varphi_2|^2 d\sigma^\varepsilon \right. \\ &\quad \left. + \varepsilon \int_{\Gamma^\varepsilon} |\varphi_1|^2 d\sigma^\varepsilon \right). \end{aligned} \quad (3.8)$$

Now, combining (3.4) and (3.8) give

$$\begin{aligned} \int_{\Omega_2^\varepsilon} |\varphi_2|^2 dx &\leq C \left(\int_{\Omega_1^\varepsilon} |\nabla \varphi_1|^2 dx + \varepsilon^2 \int_{\Omega_2^\varepsilon} |\nabla \varphi_2|^2 dx \right. \\ &\quad \left. + \varepsilon \int_{\Gamma^\varepsilon} |\varphi_1 - \varphi_2|^2 d\sigma^\varepsilon \right). \end{aligned}$$

which means that

$$\int_{\Omega_T^\varepsilon} |\varphi_2|^2 dx \leq C \|(\varphi_1, \varphi_2)\|_{V^\varepsilon}^2. \quad (3.9)$$

Observe that (3.2) yields

$$\int_{\Omega_T^\varepsilon} |\varphi_1|^2 dx \leq C \|(\varphi_1, \varphi_2)\|_{V^\varepsilon}^2. \quad (3.10)$$

Using (3.7), (3.9) and (3.10) we deduce that

$$|L^\varepsilon((\varphi_1, \varphi_2))| \leq C \|(\varphi_1, \varphi_2)\|_{V^\varepsilon}. \quad (3.11)$$

Thus, L^ε is continuous on $L^2(0, T; V^\varepsilon)$. Note that the constant C appearing in (3.7) is independent of ε .

By Lions Lemma, we conclude that there exists a unique solution $(T^\varepsilon, T_b^\varepsilon) \in L^2(0, T; V^\varepsilon)$ to the weak formulation of (2.2a)-(2.2g). Finally, putting $(\varphi_1, \varphi_2) = (T^\varepsilon, T_b^\varepsilon)$ in (2.4), using the uniform coerciveness of $a^\varepsilon(\cdot, \cdot)$, the continuity of L^ε and the Gronwall inequality yield the uniform estimate (2.5). This concludes the proof of the Theorem. \square

Remark 1. If $h = (h, h_b)$ is given in V^ε then one can easily see that $w^\varepsilon = (T^\varepsilon, T_b^\varepsilon) \in W^{1,2}(0, T; H^\varepsilon)$ and therefore $w^\varepsilon \in C(0, T; V^\varepsilon)$.

4. THE HOMOGENIZATION PROCEDURE

We shall first use the formal two-scale method (see for example A. Bensoussan & *al.*[10] and E. Sanchez-Palencia[24]) to derive the homogenized system of (2.2a)-(2.2g). To this end, let us assume the following formal expansions for the two temperatures:

$$T^\varepsilon(t, x) = T_0(t, x, y) + \varepsilon T_1(t, x, y) + \varepsilon^2 T_2(t, x, y) + \dots \quad (4.1)$$

$$T_b^\varepsilon(t, x) = T_b(t, x, y) + \varepsilon T_{b1}(t, x, y) + \varepsilon^2 T_{b2}(t, x, y) + \dots \quad (4.2)$$

where $y = x/\varepsilon$ is the microscopic variable and $T_k(\cdot, \cdot, y), T_{bk}(\cdot, \cdot, y), \dots$ ($k = 0, 1, 2, \dots$) are smooth unknown functions that are Y -periodic in the third variable y . The idea of the two-scale method is to plug the above asymptotic expansions (4.1)-(4.2) into the set of equations (2.2a)-(2.2g) and to identify powers of ε . This yields a hierarchy of initial boundary value problems for the successive terms T_k, T_{bk} .

Notation 1. In what follows, the subscript x, y on a differential operator denotes the derivative with respect to x, y respectively.

At the first step, Equation (2.2a) at ε^{-2} order and Equation (2.2c) at ε^{-1} order give

$$-\operatorname{div}_y(\alpha \nabla_y T_0) = 0 \text{ in } Q \times Y_1 \quad (4.3)$$

and

$$\alpha \nabla_y T_0 \cdot \nu = 0 \text{ on } Q \times \Sigma, \quad (4.4)$$

where ν is the unit outward normal to Σ . Testing (4.3) by $\zeta \in H_{\#}^1(Y_1)$, integrating by parts on Y_1 , taking into account (4.4) and the Y -periodicity of $\alpha \nabla_y T_0$, we get the following weak formulation:

$$\begin{cases} a_{Y_1}(T_0, \zeta) \stackrel{\text{def}}{=} \int_{Y_1} \alpha \nabla_y T_0 \cdot \nabla_y \zeta \, dy = 0 & \text{for every } \zeta \in H_{\#}^1(Y_1) / \mathbb{R}, \\ T_0 \in H_{\#}^1(Y_1) / \mathbb{R}. \end{cases}$$

The bilinear form a_{Y_1} is clearly continuous and coercive on $H_{\#}^1(Y_1) / \mathbb{R}$ and therefore standard results on uniformly elliptic equations in periodic domain (A. Bensoussan & *al.*[10]) yields that T is independent of the periodic variable y , namely there exist $T(t, x)$

$$T_0(t, x, y) = T(t, x), \quad \text{for a.e. } t \in (0, T) \text{ and } x \in \Omega.$$

Next, in the second step, Equation (2.2a) at ε^{-1} , Equation (2.2c) at ε^0 orders give the following corrector problem:

$$-\alpha \Delta_y T_1 = 0 \text{ in } Q \times Y_1, \quad (4.5)$$

$$(\alpha \nabla_y T_1) \cdot \nu = -(\alpha \nabla T) \cdot \nu \text{ on } Q \times \Sigma, \quad (4.6)$$

$$y \mapsto T_1(x, y) \text{ } Y\text{-periodic, } (t, x) \in Q. \quad (4.7)$$

The corresponding weak formulation is given by

$$(\mathcal{P}) \quad \begin{cases} a_{Y_1}(T_1, \zeta) = -a_{Y_1}(T, \zeta), \quad \forall \zeta \in H_{\#}^1(Y_1) / \mathbb{R}, \\ T_1 \in H_{\#}^1(Y_1) / \mathbb{R}. \end{cases}$$

As before, there exists a unique solution $T_1(t, x, \cdot) \in H_{\#}^1(Y_1) / \mathbb{R}$ of problem (\mathcal{P}) which can be computed as follows. Let us consider for $1 \leq i \leq d$, the following cell problems:

$$(\mathcal{P}_i) \quad \begin{cases} a_{Y_1}(\omega_i, \zeta) = -a_{Y_1}(e_i, \zeta), \quad \forall \zeta \in H_{\#}^1(Y_1) / \mathbb{R}, \\ \omega_i \in H_{\#}^1(Y_1) / \mathbb{R} \end{cases}$$

where $(e_i)_{1 \leq i \leq d}$ is the canonical basis. These problems (\mathcal{P}_i) are obtained from (\mathcal{P}) by replacing ∇T with the vector e_i , $i = 1, 2, \dots, d$. It follows

that for each i Problem (\mathcal{P}_i) admits a unique solution $\omega_i \in H_{\#}^1(Y_1)/\mathbb{R}$. Furthermore, thanks to the linearity of (\mathcal{P}) , we may write that:

$$T_1(t, x, y) = \sum_{i=1}^d \frac{\partial T}{\partial x_i}(t, x) \omega_i(y) + \tilde{u}(t, x) \quad (4.8)$$

for a.e. $(t, x, y) \in Q \times Y_1$ and where $\tilde{u}(t, x)$ is any additive constant.

At the final step, Equation (2.2a)-(2.2b) at ε^0 , Equations (2.2c)-(2.2d) at ε^1 orders yield the following initial boundary-value problem:

$$\begin{aligned} -\alpha \Delta_y T_2 &= f - \partial_t T + \alpha \operatorname{div}_y (\nabla_x T_1) + \\ &\quad \alpha \operatorname{div}_x ((\nabla_y T_1 + \nabla_x T)) \text{ in } Q \times Y_1, \end{aligned} \quad (4.9)$$

$$\partial_t T_b - \alpha_b \Delta_y T_b = f_b \text{ in } Q \times Y_2, \quad (4.10)$$

$$(\alpha \nabla_y T_2) \cdot \nu = -(\alpha \nabla_x T_1) \cdot \nu + (\alpha_b \nabla_y T_b) \cdot \nu \text{ on } Q \times \Sigma, \quad (4.11)$$

$$\alpha \nabla_y T_2 \cdot \nu = -\alpha \nabla_x T_1 \cdot \nu + \gamma (T - T_b) \text{ on } Q \times \Sigma, \quad (4.12)$$

$$y \mapsto T_2(t, x, y) \text{ } Y\text{-periodic}, \quad (4.13)$$

$$y \mapsto T_b(t, x, y) \text{ } Y\text{-periodic}. \quad (4.14)$$

The weak formulation of Equations (4.9)–(4.13) is

$$(\mathcal{P}) \begin{cases} a_{Y_1}(T_2, \zeta) = \langle F, \zeta \rangle \text{ for every } \zeta \in H_{\#}^1(Y_1)/\mathbb{R}, \\ T_2 \in H_{\#}^1(Y_1)/\mathbb{R} \end{cases}$$

where

$$\begin{aligned} \langle F, \psi \rangle &= \left(\int_{Y_1} -\zeta \, dy \right) \partial_t T + \int_{Y_1} \operatorname{div}_x (\alpha (\nabla_y T_1 + \nabla_x T)) \zeta \, dy \\ &\quad - \int_{Y_1} \alpha \nabla_x T_1 \cdot \nabla_y \zeta \, dy + \int_{Y_1} f \zeta \, dy + \int_{\Sigma} \gamma (T - T_b) \zeta \, d\sigma \end{aligned}$$

Using the divergence Theorem (as in E. Sanchez-Palencia[24]), a necessary condition for the existence of T_2 is that $\langle F, 1 \rangle = 0$, namely

$$\begin{aligned} &\int_{Y_1} (-\partial_t T + \operatorname{div}_x (\alpha (\nabla_y T_1 + \nabla_x T))) \, dy \\ &\quad + \int_{\Sigma} \gamma (T - T_b) \, d\sigma = |Y_1| f. \end{aligned} \quad (4.15)$$

Using (4.8), equation (4.15) becomes

$$|Y_1| \partial_t T - \operatorname{div} (\tilde{A} \nabla T) + \int_{\Sigma} \gamma (T - T_b) \, d\sigma = |Y_1| f, \quad (4.16)$$

where $|Y_1|$ stands for the volume of Y_1 , The matrix \tilde{A} is given by

$$\tilde{A} = (\tilde{a}_{ij})_{1 \leq i, j \leq d}, \quad \tilde{a}_{ij} = \int_{Y_1} \alpha (\nabla_y \omega_i + e_i) \cdot (\nabla_y \omega_j + e_j) dy. \quad (4.17)$$

Equation (4.16) is the so-called macroscopic equation for the temperature T . The boundary condition for T is obtained from (2.2e) at ε^0 order and it reads

$$T = 0 \text{ on } \partial\Omega. \quad (4.18)$$

Similarly, Equations (2.2f)-(2.2g) give the initial conditions for T and T_b :

$$T(0, x) = |Y_1| h(x), \quad x \in \Omega, \quad (4.19)$$

$$T_{b0}(0, x, y) = \chi_2(y) h_b(x), \quad x \in \Omega, \quad y \in Y_2 \quad (4.20)$$

Next, we proceed further and focus our attention on fluid temperature $T_b(t, x, y)$. From (4.11) and (4.12) it is easily seen that

$$\alpha_b \nabla_y T_b \cdot \nu = \gamma (T - T_b) \text{ on } Q \times \Sigma. \quad (4.21)$$

It is easily shown that (4.10), (4.21) and (4.20) admits a unique weak solution $T_b \in H_{\#}^1(Y_2)$. Moreover, denoting

$$b(\zeta, \eta) = \int_{Y_2} \alpha_b \nabla_y \zeta \cdot \nabla_y \eta dy + \int_{\Sigma} \gamma \zeta \eta d\sigma, \quad \zeta, \eta \in H^1(Y_2)$$

and $\mathcal{B} : H^1(Y_2) \longrightarrow H^1(Y_2)$ defined by $\langle \mathcal{B}(\zeta), \eta \rangle = b(\zeta, \eta)$, and applying (as in U. Hornung[19]) the Duhamel's principle to equations (4.10), (4.21) and (4.20), The leading term T_b can be thus decomposed as the sum of three terms:

$$\begin{aligned} T_b(\tau, x, y) &= T_{bi}(\tau, x, y) + \int_0^\tau \partial_\tau \omega(\tau - t, y) T(t, x) dt \\ &\quad + \int_0^\tau \partial_\tau \mu(\tau - t, y) f_b(t, x) dt \end{aligned} \quad (4.22)$$

where T_{bi} is the evolution of the initial temperature h_b . It is given by $T_{bi}(t, x, y) = e^{-t\mathcal{B}} h_b(x, y) = \theta(t, y) h_b(x)$ where $\theta(t, y)$ is the unique of the weak solution of the cell problem:

$$\begin{aligned} \partial_t \theta - \alpha_b \Delta_y \theta &= 0 \text{ in } (0, T) \times Y_2, \\ \alpha_b \nabla_y \theta \cdot \nu + \gamma \theta &= 0 \text{ on } (0, T) \times \Sigma, \\ y &\longmapsto \theta(t, y) \text{ } Y\text{-periodic}, \\ \theta(0, y) &= 1 \text{ in } \Omega \times Y_2. \end{aligned}$$

On the other hand ω , σ are respectively the unique weak solutions of the following cell problems:

$$\begin{aligned} \partial_t \omega - \alpha_b \Delta_y \omega &= 0 \text{ in } (0, T) \times Y_2, \\ \alpha_b \nabla_y \omega \cdot \nu + \gamma \omega &= \gamma \text{ on } (0, T) \times \Sigma, \\ y &\longmapsto \omega(t, y) \text{ } Y\text{-periodic,} \\ \omega(0, y) &= 0 \text{ in } \Omega \times Y_2, \end{aligned}$$

and

$$\begin{aligned} \partial_t \sigma - \alpha_b \Delta_y \sigma &= 1 \text{ in } (0, T) \times Y_2, \\ \alpha_b \nabla_y \sigma \cdot \nu + \gamma \sigma &= 0 \text{ on } (0, T) \times \Sigma, \\ y &\longmapsto \sigma(t, y) \text{ } Y\text{-periodic,} \\ \sigma(0, y) &= 0 \text{ in } \Omega \times Y_2. \end{aligned}$$

Inserting (4.22) into (4.16) we get the homogenized integro-differential equation of Barbashin type for the temperature T (see (2.6)-(2.8)):

$$\begin{aligned} \partial_t T - \int_0^t \mathcal{H}(t-\tau) T(\tau) d\tau - \operatorname{div} \left(\tilde{A} \nabla T \right) + \tilde{\gamma} T &= \mathcal{F} \text{ in } Q, \\ T &= 0 \text{ on } S, \\ T(0, x) &= |Y_1| h(x), \quad x \in \Omega \end{aligned}$$

where

$$\tilde{\gamma} = \frac{1}{|Y_1|} \int_{\Sigma} \gamma d\sigma \quad (4.23)$$

and, where \mathcal{H} and \mathcal{F} are given by

$$\mathcal{H}(\tau, x) = \frac{1}{|Y_1|} \int_{\Sigma} \gamma(y) \partial_t \omega(\tau, y) d\sigma, \quad (4.24)$$

$$\begin{aligned} \mathcal{F}(\tau, x) &= f(x) + \frac{1}{|Y_1|} \left(\int_{\Sigma} \gamma(y) T_{bi}(\tau, x, y) d\sigma + \right. \\ &\quad \left. \int_0^{\tau} \int_{\Sigma} \partial_{\tau} \sigma(\tau - t, y) f_b(\tau, y) d\sigma dt \right). \end{aligned} \quad (4.25)$$

5. PROOF OF THEOREM 2

In this section, we shall derive the homogenized system (2.6)-(2.8). To do so, we shall use the two-scale convergence technique that we recall hereafter.

We shall first begin with some notations. We define $\mathcal{C}_{\#}(Y)$ to be the space of all continuous functions on \mathbb{R}^d which are Y -periodic. Let $\mathcal{C}_{\#}^{\infty}(Y) = \mathcal{C}^{\infty}(\mathbb{R}^d) \cap \mathcal{C}_{\#}(Y)$ and let $L_{\#}^2(Y)$ (resp. $L_{\#}^2(Y_m)$, $m = 1, 2$) to be the space of all functions belonging to $L_{\text{loc}}^2(\mathbb{R}^d)$ (resp. $L_{\text{loc}}^2(Z_m)$)

which are Y -periodic, and $H_{\#}^1(Y)$ (resp. $H_{\#}^1(Y_m)$) to be the space of those functions together with their derivatives belonging to $L_{\#}^2(Y)$ (resp. $L_{\#}^2(Z_m)$).

Now, we recall the definition and main results of the two-scale convergence method. For more details, we refer the reader to G. Allaire[5].

Definition 1. A sequence (ϑ^ε) in $L^2(\Omega)$ two-scale converges to $\vartheta \in L^2(\Omega \times Y)$ (we write $\vartheta^\varepsilon \xrightarrow{2-s} \vartheta$) if, for any admissible test function $\varphi \in L^2(\Omega; \mathcal{C}_{\#}(Y))$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \vartheta^\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega \times Y} \vartheta(x, y) \varphi(x, y) dx dy.$$

Theorem 3. Let (ϑ^ε) be a sequence of functions in $L^2(\Omega)$. Assume that (ϑ^ε) is uniformly bounded. Then, there exist $\vartheta \in L^2(\Omega \times Y)$ and a subsequence of (ϑ^ε) which two-scale converges to T_b .

Theorem 4. Let (ϑ^ε) be a uniformly bounded sequence in $H^1(\Omega)$ (resp. $H_0^1(\Omega)$). Then, up to a subsequence, there exist $\vartheta \in H^1(\Omega)$ (resp. $H_0^1(\Omega)$) and $\vartheta_0 \in L^2(\Omega; H_{\#}^1(Y)/\mathbb{R})$ such that

$$\vartheta^\varepsilon \xrightarrow{2-s} \vartheta; \quad \nabla \vartheta^\varepsilon \xrightarrow{2-s} \nabla \vartheta + \nabla_y \vartheta_0.$$

The following result will be of use, see G. Allaire & al.[6, Proposition 2.6].

Theorem 5. Let (ϑ^ε) be a sequence of functions in $H^1(\Omega)$ such that

$$\|\vartheta^\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla \vartheta^\varepsilon\|_{L^2(\Omega)^3} \leq C.$$

Then there exist $\vartheta \in L^2(\Omega; H_{\#}^1(Y))$ and a subsequence of (ϑ^ε) , still denoted by (ϑ^ε) , such that

$$\vartheta^\varepsilon \xrightarrow{2-s} \vartheta, \quad \varepsilon \nabla \vartheta^\varepsilon \xrightarrow{2-s} \nabla_y \vartheta$$

and for every $\varphi \in \mathcal{D}(\Omega; \mathcal{C}_{\#}(Y))$ we have:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma^\varepsilon} \vartheta^\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) d\sigma^\varepsilon = \int_{\Omega \times \Sigma} \vartheta(x, y) \varphi(x, y) dx d\sigma.$$

The notion of two-scale convergence can easily be extended to time-dependent sequences without affecting the results stated above, namely Theorems 3, 4 and 5. According to G.W. Clark and R. Showalter[15], we give the following:

Definition 2. We say that a sequence (ϑ^ε) in $L^2(Q)$ two-scale converges to $\vartheta \in L^2(Q \times Y)$ (we write $\vartheta^\varepsilon \xrightarrow{2-s} \vartheta$) if, for any test function

$\varphi \in L^2(Q; \mathcal{C}_\#(Y))$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_Q \vartheta^\varepsilon(t, x) \varphi\left(t, x, \frac{x}{\varepsilon}\right) dt dx = \int_{Q \times Y} \vartheta(t, x, y) \varphi(t, x, y) dt dx dy.$$

Remark 2. If (ϑ^ε) is a uniformly bounded sequence in $L^2(Q)$, then there exists $\vartheta \in L^2(Q)$ such that, up to a subsequence, $\vartheta^\varepsilon \xrightarrow{2-s} \vartheta$ in the sense of Def. 2. Moreover, if (ϑ^ε) is uniformly bounded in $L^2(0, T; H^1(\Omega))$, then up to a subsequence, there exist $\vartheta \in L^2(0, T; H^1(\Omega))$ and $\vartheta_0 \in L^2(Q; H_\#^1(Y)/\mathbb{R})$ such that $\vartheta^\varepsilon \xrightarrow{2-s} \vartheta$ and $\nabla \vartheta^\varepsilon \xrightarrow{2-s} \nabla \vartheta + \nabla_y \vartheta_0$. On the other hand, if a sequence (ϑ^ε) is such that

$$\|\vartheta^\varepsilon\|_{L^2(Q)} + \varepsilon \|\nabla \vartheta^\varepsilon\|_{L^2(Q)} \leq C,$$

then, up to a subsequence, there exists $\vartheta \in L^2(0, T; H_\#^1(Y))$ such that $\vartheta^\varepsilon \xrightarrow{2-s} \vartheta$ and $\varepsilon \nabla \vartheta^\varepsilon \xrightarrow{2-s} \nabla_y \vartheta$. Furthermore, for every $\varphi \in \mathcal{D}(Q; \mathcal{C}_\#(Y))$ we have:

$$\lim_{\varepsilon \rightarrow 0} \int_{S^\varepsilon} \vartheta^\varepsilon(t, x) \varphi\left(t, x, \frac{x}{\varepsilon}\right) d\sigma^\varepsilon = \int_{Q \times \Sigma} \vartheta(t, x, y) \varphi(t, x, y) dx ds$$

where ds denotes the surface measure on Σ .

Next we focus our attention on the two-scale convergence process, that is deriving the two-scale homogenized system by employing the above compactness results: Theorems 3-5 and Remark 2. To do this, let us choose the following test functions: Let

$$\varphi_1 \in W^{1,2}(0, T; \mathcal{D}(\Omega)), \quad \psi \in W^{1,2}(0, T; \mathcal{D}(\Omega; \mathcal{C}_\#(Y)))$$

and

$$\varphi_2 \in W^{1,2}(0, T; \mathcal{D}(\Omega; \mathcal{C}_\#(Y)))$$

with

$$\varphi_1(T, \cdot, \cdot) = \varphi_2(T, \cdot, \cdot) = 0.$$

Set

$$\varphi^\varepsilon(t, x) = \left(\varphi_1(t, x) + \varepsilon \psi\left(t, x, \frac{x}{\varepsilon}\right), \varphi_2\left(t, x, \frac{x}{\varepsilon}\right) \right).$$

Taking $\varphi = \varphi^\varepsilon$ as a test function in (2.4), we get

$$\begin{aligned}
& - \int_{Q_1^\varepsilon} T^\varepsilon \partial_t \varphi_1 - \int_{Q_2^\varepsilon} T_b^\varepsilon \partial_t \varphi_2 + \int_{Q_1^\varepsilon} \alpha \nabla T^\varepsilon (\nabla \varphi_1 + (\nabla_y \psi)^\varepsilon) + \\
& \varepsilon \int_{Q_2^\varepsilon} \alpha_b \nabla T_b^\varepsilon (\nabla_y \varphi_2)^\varepsilon + \varepsilon \int_{S^\varepsilon} \gamma (T^\varepsilon - T_b^\varepsilon) (\varphi_1 - \varphi_2) \\
& = \int_{Q_1^\varepsilon} f \varphi_1 + \int_{Q_2^\varepsilon} f_b \varphi_2 + \int_{\Omega_1^\varepsilon} h \varphi_1 (0) + \int_{\Omega_2^\varepsilon} h_b \varphi_2 (0) + \varepsilon K^\varepsilon \quad (5.1)
\end{aligned}$$

where

$$\begin{aligned}
K^\varepsilon &= O(\varepsilon) = \int_{Q_1^\varepsilon} T^\varepsilon \partial_t \psi \, dt \, dx - \int_{Q_1^\varepsilon} \alpha \nabla T^\varepsilon (\nabla_x \psi)^\varepsilon \, dt \, dx \\
&+ \int_{Q_1^\varepsilon} f \psi \, dt \, dx + \int_{\Omega_1^\varepsilon} h \psi (0) \, dx + \varepsilon \int_{S^\varepsilon} \gamma (T^\varepsilon - T_b^\varepsilon) \psi \, dt \, d\sigma^\varepsilon.
\end{aligned}$$

Now, thanks to the assumptions (2.3) and to the a priori estimates (2.5), using Theorems 3-5 and Remark 2, we have up to a subsequence, the following two scale convergences:

$$\begin{aligned}
& \chi_1 T^\varepsilon \xrightarrow{2-s} \chi_1 T, \quad \chi_2 T_b^\varepsilon \xrightarrow{2-s} \chi_2 T_b, \\
& \chi_1 \nabla T^\varepsilon \xrightarrow{2-s} \chi_1 (\nabla T + \nabla_y T_1), \quad \varepsilon \chi_2 \nabla T_b^\varepsilon \xrightarrow{2-s} \chi_2 \nabla_y T_b, \\
& \lim_{\varepsilon \rightarrow 0} \int_{S^\varepsilon} T^\varepsilon \varphi \left(t, x, \frac{x}{\varepsilon} \right) \, d\sigma^\varepsilon = \int_{Q \times \Sigma} T \varphi(t, x, y) \, dx \, ds, \\
& \lim_{\varepsilon \rightarrow 0} \int_{S^\varepsilon} T_b^\varepsilon \varphi \left(t, x, \frac{x}{\varepsilon} \right) \, d\sigma^\varepsilon = \int_{Q \times \Sigma} T_b \varphi(t, x, y) \, dx \, ds,
\end{aligned}$$

where $T \in L^2(0, T; H_0^1(\Omega))$, $T_b \in L^2(Q; H_\#^1(Y))$ and $T_1 \in L^2(Q; H_\#^1(Y)/\mathbb{R})$. Now, passing to the limit in (5.1) and taking into account the above limits yield the two scale system:

$$\begin{aligned}
& - \int_{Q \times Y_1} T \partial_t \varphi_1 - \int_{Q \times Y_2} T_b \partial_t \varphi_2 + \int_{Q \times Y_1} \alpha (\nabla T + \nabla_y T_1) (\nabla \varphi_1 + \nabla_y \psi) \\
& + \int_{Q \times Y_2} \alpha_b \nabla_y T_b \nabla_y \varphi_2 + \int_{Q \times \Sigma} \gamma (T - T_b) (\varphi_1 - \varphi_2) \\
& = \int_{Q \times Y_1} f \varphi_1 + \int_{Q \times Y_2} f_b \varphi_2 + \int_{\Omega \times Y_1} h \varphi_1 + \int_{\Omega \times Y_2} h_b \varphi_2 \quad (5.2)
\end{aligned}$$

Now, integration by parts in (5.2) yields

$$|Y_1| \partial_t T - \alpha \operatorname{div} \left(\int_{Y_1} (\nabla T + \nabla_y T_1) \right) + \int_{\Sigma} \gamma (T - T_b) = |Y_1| f \text{ in } Q; \quad (5.3)$$

$$\partial_t T_b - \alpha_b \Delta_y T_b = |Y_2| f_b \text{ in } Q \times Y_2; \quad (5.4)$$

$$-\alpha \operatorname{div}_y (\nabla T + \nabla_y T_1) = 0 \text{ in } Q \times Y_2; \quad (5.5)$$

$$\alpha (\nabla T + \nabla_y T_1) \cdot \nu = 0 \text{ on } Q \times \Sigma; \quad (5.6)$$

$$\alpha_b \nabla_y T_b \cdot \nu = \gamma (T - T_b) \text{ on } Q \times \Sigma; \quad (5.7)$$

$$y \longmapsto T_1 \text{ } Y \text{ - periodic}; \quad (5.8)$$

$$y \longmapsto T_b \text{ } Y \text{ - periodic}; \quad (5.9)$$

$$T(0) = |Y_1| h \text{ in } Q; \quad (5.10)$$

$$T(0) = \chi_2 h_b \text{ in } Q \times Y_2. \quad (5.11)$$

$$T = 0 \text{ on } \Gamma \quad (5.12)$$

Finally we observe that the equations of the system (5.3)-(5.12) are exactly and respectively (4.15), (4.10), (4.5), (4.6), (4.21), (4.7), (4.14), (4.19), (4.20) and (4.18). Therefore we have recovered the same process done in the previous section and consequently the formal asymptotic expansion method used to construct the homogenized problem (2.6)-(2.8) is justified. Thus Theorem 2 is proved.

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